Optional decompositions under constraints

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Abstract

Motivated by a hedging problem in mathematical finance, El Karoui and Quenez [7] and Kramkov [14] have developed optional versions of the Doob-Meyer decomposition which hold simultaneously for all equivalent martingale measures. We investigate the general structure of such optional decompositions, both in additive and in multiplicative form, and under constraints corresponding to different classes of equivalent measures. As an application, we extend results of Karatzas and Cvitanić [3] on hedging problems with constrained portfolios.

1 Introduction

Let V be a non-negative supermartingale on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. The Doob-Meyer decomposition implies that V can be represented in the form

$$V = V_0 + M - C,$$

where M is a local martingale and C is an increasing optional process. Such an optional decomposition is in general not unique. In fact, the Doob-Meyer decomposition asserts existence and uniqueness under the additional requirement that C is predictable.

Now suppose that the supermartingale property of V holds simultaneously for all probability measures $Q \sim P$ such that a given semimartingale X is a local martingale under Q. Denote by $\mathcal{P}(X)$ the class of these measures. In this case and under the assumption that X is locally bounded, Kramkov [14] has shown that an optional decomposition of the form

$$V = V_0 + \int H dX - C \tag{1.1}$$

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holds where C is an increasing optional process. The stochastic integral is a local martingale for any $Q \in \mathcal{P}(X)$. Thus, we have an optional decomposition which is valid simultaneously for all measures in the class $\mathcal{P}(X)$. The local boundedness assumption has been removed in Föllmer and Kabanov [10]; see also Delbaen and Schachermayer [4], Kramkov [13]. An optional decomposition was first proved by El Karoui and Quenez [7] in the special case where X is a diffusion process, and where the process V is of the form

$$V_t = \operatorname*{ess\,sup}_{Q \in \mathcal{P}(X)} E_Q[f_T | \mathcal{F}_t], \quad 0 \le t \le T,$$

for some $\mathcal{F}_T$-measurable random variable f_T. In terms of mathematical finance, V is the value process associated to the problem of hedging a contingent claim f_T with complete safety in an incomplete situation where the equivalent martingale measure is not unique. From this point of view, an optional decomposition of the form (1.1) provides a hedging strategy H which covers perfectly the given claim, and at the same time yields cumulative side payments described by the increasing process C. This interpretation suggests to investigate the structure of optional decompositions under additional constraints on the integrand H.

In this paper we derive optional decompositions in the following general setting. We prescribe a convex class $\mathcal{S}$ of semimartingales, for example a class of stochastic integrals whose integrands satisfy certain convex constraints. We look for a decomposition of the form

$$V = V_0 + S - C,$$

where $S \in \mathcal{S}$ and C is an increasing optional process. Our criterion for the existence of such an optional decomposition takes the following form: The process

$$V - A^{\mathcal{S}}(Q)$$

is a supermartingale under any measure Q in a certain class $\mathcal{P}(\mathcal{S})$, where $A^{\mathcal{S}}(Q)$ is an increasing predictable process depending only on Q and $\mathcal{S}$. If $\mathcal{S}$ is a linear space then $\mathcal{P}(\mathcal{S})$ is the class of all equivalent local martingale measures for $\mathcal{S}$. If $\mathcal{S}$ is a cone then $\mathcal{P}(\mathcal{S})$ is the class of all equivalent local supermartingale measures. In both cases, the process $A^{\mathcal{S}}(Q)$ is equal to 0. If $\mathcal{S}$ is a class of stochastic integrals of X where the integrands satisfy certain convex constraints then these constraints are incorporated in the process $A^{\mathcal{S}}(Q)$.

If constraints are formulated not in terms of the integrands H but in terms of the proportions $H^i X^i / V$, one is led to an analogous multiplicative decomposition:

$$V = V_0 \mathcal{E}(S - C),$$

where $S \in \mathcal{S}$, C is an increasing optional process, and $\mathcal{E}$ denotes the Doléans-Dade exponential. Here our criterion says that the process

$$V / \mathcal{E}(A^{\mathcal{S}}(Q))$$

is a supermartingale under any $Q \in \mathcal{P}(\mathcal{S})$. This leads to extensions of various results on hedging under convex constraints; see, e.g., Karatzas and Cvitanić [3].

For the theory of stochastic integration we refer to Dellacherie and Meyer [6], Protter [18], and Jacod and Shiryaev [11]. The stochastic integral of a predictable process H with respect to a semimartingale X will be denoted as $\int HdX$ or $H \bullet X$. Let $\mathbf{L}(X)$ denote the space of all predictable processes integrable with respect to X. A process $H \in \mathbf{L}(X)$ will be called (locally) *admissible* if $H \bullet X$ is (locally) bounded from below. The classes of admissible and locally admissible integrands are denoted as $\mathbf{L}^a(X)$ and $\mathbf{L}^a_{loc}(X)$. The *Émery distance* between two semimartingales X and Y is defined as

$$D(X,Y) = \sup_{|H|\le 1} \left(\sum_{n\ge 1} 2^{-n} E\left[\min(|\ (H \bullet (X-Y)_n\ |, 1)\right] \right),$$

where the supremum is taken over the set of all predictable processes H bounded by 1. For this metric the space of semimartingales is complete, see Émery [9]. The corresponding topology is called the *semimartingale* or *Émery* topology. If X is a semimartingale then the space $\mathbf{L}(X)$ is complete with respect to the metric

$$d_X(H,G) = D(H \bullet X, G \bullet X); \tag{1.2}$$

see Mémin [17].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ denote a filtered probability space which satisfies the "usual" conditions. Except for processes which appear as integrands of stochastic integrals, all processes considered in the sequel are assumed to be real-valued, to have right-continuous paths with left limits, and to be adapted with respect to the given filtration; in particular they are all optional. For two such processes X and Y, the relation $X \prec Y$ means that $Y - X$ is an increasing process.

2 The upper variation process for a family of semimartingales

Let $\mathcal{S}$ be a family of semimartingales which are locally bounded from below with initial value $S_0 = 0$. We assume that $\mathcal{S}$ contains the constant process $S \equiv 0$.

Let us introduce the class $\mathcal{P}(\mathcal{S})$ of all probability measures $Q \sim P$ such that any $S \in \mathcal{S}$ is a special semimartingale under Q, and such that there is an upper bound for all the increasing predictable processes arising in the Doob-Meyer decomposition of the special semimartingales $S \in \mathcal{S}$ under Q. In other words:

Definition 2.1 We denote by $\mathcal{P}(\mathcal{S})$ the class of all probability measures $Q \sim P$ with the following property: There exists an increasing predictable process A (depending on Q and $\mathcal{S}$) such that $S - A$ is a local supermartingale under Q for any $S \in \mathcal{S}$, i.e.,

$$A^S(Q) \prec A \qquad \forall S \in \mathcal{S}, \tag{2.1}$$

where $A^S(Q)$ denotes the compensator of S under Q. An increasing predictable process $A^{\mathcal{S}}(Q)$ will be called an *upper variation process* of $\mathcal{S}$ under Q if it satisfies condition (2.1) and is *minimal* with respect to this property, i.e.,

$$A^{\mathcal{S}}(Q) \prec A$$

for any predictable increasing process A which satisfies (2.1).

Example 2.1 Let $\mathcal{S}$ be a *linear* family of locally bounded processes. Then a measure $Q \sim P$ belongs to $\mathcal{P}(\mathcal{S})$ if and only if each $S \in \mathcal{S}$ is a *local martingale* under Q, i.e.,

$$\mathcal{P}(\mathcal{S}) = \{Q \sim P| \quad Q \text{ is a local martingale measure for } \mathcal{S}\ \}.$$

In this case, the upper variation process is given by $A^{\mathcal{S}}(Q) \equiv 0$.

Example 2.2 Let $\mathcal{S}$ be a *cone* of processes which are locally bounded from below. Then a measure $Q \sim P$ belongs to $\mathcal{P}(\mathcal{S})$ if and only if each $S \in \mathcal{S}$ is a *local supermartingale* under Q, i.e.,

$$\mathcal{P}(\mathcal{S}) = \{Q \sim P| \quad Q \text{ is a local supermartingale measure for } \mathcal{S}\ \}.$$

Here again, the upper variation process is given by $A^{\mathcal{S}}(Q) \equiv 0$.

Definition 2.2 The family $\mathcal{S}$ will be called *predictably convex* if for $S^i \in \mathcal{S}$ $(i = 1, 2)$ and for any predictable process h such that $0 \leq h \leq 1$ we have

$$h \bullet S^1 + (1-h) \bullet S^2 \in \mathcal{S}.$$

¿From now on we assume that $\mathcal{S}$ is predictably convex. Under this assumption, we show that the upper variation process exists for any $Q \in \mathcal{P}(\mathcal{S})$, and that it can be constructed as the essential supremum of the family of compensators under Q.

Lemma 2.1 *A probability measure $Q \sim P$ belongs to $\mathcal{P}(\mathcal{S})$ iff any $S \in \mathcal{S}$ is a special semimartingale under Q and*

$$\operatorname*{ess\,sup}_{S \in \mathcal{S}} A^S(Q)_t < +\infty \tag{2.2}$$

a.s. for all $t \geq 0$. In this case the upper variation process exists and is uniquely determined by the equations

$$\begin{aligned} A^{\mathcal{S}}(Q)_\tau &= \operatorname*{ess\,sup}_{S \in \mathcal{S}} A^S(Q)_\tau, \\ E[A^{\mathcal{S}}(Q)_\tau] &= \sup_{S \in \mathcal{S}} E[A^S(Q)_\tau] \end{aligned}$$

for any stopping time τ. Moreover, there exists a sequence $S^n \in \mathcal{S}$ such that the compensators $A^n = A^{S^n}(Q)$ satisfy $A^n \prec A^{n+1}$ and

$$\lim_{n\to\infty} \sup_{0 \leq t \leq \tau} \left(A^{\mathcal{S}}(Q)_t - A^n_t \right) = 0 \quad a.s.$$

for any stopping time τ such that $A^{\mathcal{S}}(Q)_\tau < +\infty$ a.s..

Proof The necessity of (2.2) is obvious. To prove sufficiency we assume first that $\mathcal{S}$ consists of predictable processes of bounded variation. In this case, the set $\mathcal{P}(\mathcal{S})$ is either empty or it contains all $Q \sim P$. For A and B in $\mathcal{S}$ consider the process

$$C = \frac{1}{2}\left(A + B + \operatorname{Var}(A - B)\right),$$

where

$$\mathrm{Var}(A-B) = (A-B)^+ + (A-B)^-$$

is defined in terms of the Hahn decomposition of $A - B$. We have $A \prec C$ and $B \prec C$, and C belongs to $\mathcal{S}$ since $\mathcal{S}$ is predictably convex. Thus $\mathcal{S}$ is upwards directed. By this property and standard diagonalization arguments, one can construct a sequence $(C^n)_{n\geq 1}$ in $\mathcal{S}$ such that $C^n \prec C^{n+1}$ and

$$\lim_{n\to\infty} C_t^n = \operatorname*{ess\,sup}_{S\in\mathcal{S}} S_t$$

for all rational $t \geq 0$. Moreover, this convergence is uniform on any interval $[0,\tau]$ such that $\operatorname{ess\,sup}_{S\in\mathcal{S}} S_\tau < +\infty$. The resulting limit process, denoted by $\operatorname{ess\,sup}_{S\in\mathcal{S}} S$, is increasing and predictable, and it dominates any $A \in \mathcal{S}$. Thus, $\operatorname{ess\,sup}_{S\in\mathcal{S}} S$ is the upper variation process with respect to any $Q \sim P$.

The general case follows if we apply the same argument to the space of compensators of $S \in \mathcal{S}$ with respect to $Q \in \mathcal{P}(\mathcal{S})$. $\square$

Example 2.3 Let X be a semimartingale, and let $\overline{G} \geq 0$ and $\underline{G} \leq 0$ belong to the space $\mathbf{L}^a_{loc}(X)$ of locally admissible integrands for X. In other words, the stochastic integrals $\overline{G} \bullet X$ and $\underline{G} \bullet X$ are well defined and are locally bounded from below. We denote by

$$\mathcal{H} = \left\{ H : \quad \underline{G} \leq H \leq \overline{G} \right\}$$

the family of predictable processes bounded from above by $\overline{G}$ and from below by $\underline{G}$. All stochastic integrals $H \bullet X$ for $H \in \mathcal{H}$ are locally bounded from below, the class

$$\mathcal{S} = \{ H \bullet X : \quad H \in \mathcal{H} \}$$

is predictably convex, and so $\mathcal{S}$ satisfies our assumptions above. Let $Q \sim P$ be such that X is a special semimartingale with respect to Q, and denote by

$$X = M + A$$

the canonical decomposition, where M is a local martingale under Q and A is a predictable process of bounded variation. The compensator of any process $S = H \bullet X \in \mathcal{S}$ has the form $H \bullet A$ with $H \in \mathcal{H}$, and we have the estimate

$$H \bullet A = H \bullet A^+ - H \bullet A^- \leq \overline{G} \bullet A^+ - \underline{G} \bullet A^-.$$

On the other hand, the equality is achieved for $H = h\overline{G} + (1-h)\underline{G}$, where $h = dA^+/dA$. Thus, Lemma 2.1 implies that the upper variation process $A^{\mathcal{S}}(Q)$ of $\mathcal{S}$ under Q is given by

$$A^{\mathcal{S}}(Q)_t = \int_0^t \overline{G}_s dA_s^+ - \int_0^t \underline{G}_s dA_s^-, \quad t \geq 0.$$

In particular we have shown that the set $\mathcal{P}(\mathcal{S})$ contains all probability measures $Q \sim P$ such that X is a special semimartingale under Q.

3 A decomposition theorem

As in the previous section we consider a family $\mathcal{S}$ of semimartingales which is predictably convex and contains the process 0, and such that all processes $S \in \mathcal{S}$ are locally bounded from below with initial value $S_0 = 0$. Moreover we assume that $\mathcal{P}(\mathcal{S}) \neq \emptyset$ and that the set $\mathcal{S}$ has the following closure property:

Assumption 3.1 If (S^n) is a sequence in $\mathcal{S}$ which is uniformly bounded from below and converges in the semimartingale topology to S then we have $S \in \mathcal{S}$.

If $\mathcal{S}$ is a set of stochastic integrals as in Example 2.3 then this closure property will follow from a theorem of Mémin [17]. We consider this case in more detail in the next section.

Theorem 3.1 *Let V be a process which is locally bounded from below. Then the following statements are equivalent:*

(i) V admits a decomposition

$$V = V_0 + S - C,$$

where $S \in \mathcal{S}$, and C is an increasing process;

(ii) for all $Q \in \mathcal{P}(\mathcal{S})$ the process $V - A^{\mathcal{S}}(Q)$ is a local supermartingale under Q.

Remarks 3.1 1) The theorem can also be stated without Assumption 3.1 and without the assumption that $\mathcal{S}$ is predictably convex. In this case the equivalence holds if the process S in decomposition (i) is assumed to be in the minimal class $\hat{\mathcal{S}} \supseteq \mathcal{S}$ of semimartingales such that $\hat{\mathcal{S}}$ is predictably convex and satisfies Assumption 3.1.

2) Condition (ii) means that the process $A^V(Q)$ in the canonical decomposition $V = M + A^V(Q)$ of the special semimartingale V under Q is dominated by $A^{\mathcal{S}}(Q)$, i.e., $A^V(Q) \prec A^{\mathcal{S}}(Q)$.

The proof of Theorem 3.1 will be given in Section 6. We conclude this section with a multiplicative version of Theorem 3.1 which will be useful in the next section devoted to the application to portfolio strategies under convex constraints. Let $\mathcal{E}(X)$ denote the Doléan-Dade exponential of a semimartingale X:

$$\mathcal{E}(X) = e^{X - X_0 - \langle X^c \rangle} \prod_{s \leq \cdot} (1 + \Delta X_S)\, e^{-\Delta X_s},$$

where $\langle X^c \rangle$ denotes the quadratic variation of the continuous martingale part of X. Recall that $\mathcal{E}(X)$ is a solution of the following stochastic differential equation:

$$Z = 1 + Z_- \bullet X, \quad Z_0 = 1.$$

Moreover any solution of this equation coincides with $\mathcal{E}(X)$ on the set $\{(\omega, t) : \mathcal{E}(X)_- \neq 0\}$.

Corollary 3.1 *Let V be a nonnegative process. Under the assumptions of Theorem 3.1 the following statements are equivalent:*

(*i*) *V admits a decomposition*

$$V = V_0 \mathcal{E}(S - C),$$

where $S \in \mathcal{S}$, *and* C *is an increasing process;*

(*ii*) *for all* $Q \in \mathcal{P}(\mathcal{S})$ *the process* $V/\mathcal{E}(A^{\mathcal{S}}(Q))$ *is a supermartingale under* Q.

Proof Hereafter we assume that $V_0 = 1$. Let us recall the following formula:

$$\mathcal{E}(X)\mathcal{E}(A) = \mathcal{E}(X + A + [X, A]) = \mathcal{E}(A + (1 + \Delta A) \bullet X)), \tag{3.1}$$

which holds for any semimartingale X with initial value $X_0 = 0$ and for any predictable process A of bounded variation.

(*i*) $\Rightarrow$ (*ii*) Let $Q \in \mathcal{P}(\mathcal{S})$. Define a semimartingale X by the following formula:

$$X = \frac{1}{1 + \Delta A^{\mathcal{S}}(Q)} \bullet \left(S - C - A^{\mathcal{S}}(Q)\right).$$

Accounting for (3.1) we get

$$\mathcal{E}(X)\mathcal{E}(A^{\mathcal{S}}(Q)) = \mathcal{E}(S - C) = V.$$

¿From the definition of $A^{\mathcal{S}}(Q)$ we deduce that X is a local supermartingale under Q. Moreover $\mathcal{E}(X) \geq 0$, because $V \geq 0$. It follows that

$$\mathcal{E}(X) = V/A^{\mathcal{S}}(Q)$$

is a supermartingale under Q.

(*ii*) $\Rightarrow$ (*i*) It is sufficient to prove the existence of a decomposition (*i*) on any interval $[0, \tau_n]$, $n \geq 1$, where

$$\tau_n = \inf\left\{t \geq 0 : \quad V_t \leq \frac{1}{n}\right\}.$$

Hereafter we assume that all processes are defined on the set $\Gamma = \cup_{n\geq 1}[0, \tau_n] = \{(\omega, t) : \ V_- > 0\}$.

Define

$$R_t = \int_0^t \frac{dV_s}{V_{s-}}.$$

We have $V = \mathcal{E}(R)$ and according to Theorem 3.1 the proof will follow if $R - A^{\mathcal{S}}(Q)$ is a local supermartingale under any $Q \in \mathcal{P}(\mathcal{S})$.

Let us fix $Q \in \mathcal{P}(\mathcal{S})$ and define

$$X = \frac{1}{1 + \Delta A^{\mathcal{S}}(Q)} \bullet \left(R - A^{\mathcal{S}}(Q)\right).$$

¿From (3.1) we deduce that

$$\mathcal{E}(X)\mathcal{E}(A^{\mathcal{S}}(Q)) = \mathcal{E}(R) = V.$$

By assumption $\mathcal{E}(X)$ is a local supermartingale under Q and, since $\mathcal{E}(X)_- > 0$ on Γ we get that X is also a local supermartingale under Q. This implies the desired supermartingale property for

$$R - A^{\mathcal{S}}(Q) = \left(1 + \Delta A^{\mathcal{S}}(Q)\right) \bullet X.$$

□

4 Constrained portfolios

Let us consider a model of a security market which consists of $d+1$ assets: one bond and d stocks. Hereafter we suppose that the bond is chosen as a numeraire and denote by $X = (X^i)_{1 \le i \le d}$ the discounted price process of the stocks.

A portfolio Π is defined as a triple (v, H, C), where the constant v is the initial value of the portfolio, $H = (H^i)_{1 \le i \le d}$ is a predictable X-integrable process specifying the amount of each asset held in the portfolio, and $C = (C_t)_{t \ge 0}$ is an increasing process of accumulated consumption. The value process $V = (V_t)_{t \ge 0}$ of such a portfolio Π is given by

$$V_t = v + \int_0^t H_s dX_s - C_t, \qquad t \ge 0. \tag{4.1}$$

The condition $C \equiv 0$ means that the portfolio Π is self-financing.

Let now $\mathcal{H} \subseteq \mathbf{L}^a_{loc}(X)$ be a family of locally admissible integrands for X. We assume that $\mathcal{H}$ contains $H \equiv 0$, is closed in $\mathbf{L}^a_{loc}(X)$ with respect to the distance d_X defined in (1.2), and is convex in the following sense: for any H and G in $\mathcal{H}$ and any predictable process $0 \le h \le 1$ the process $hH + (1-h)G$ belongs to $\mathcal{H}$. A portfolio $\Pi = (v, H, C)$ is called *$\mathcal{H}$-constrained* if $H \in \mathcal{H}$.

Example 4.1 The role of $\mathcal{H}$ is to model various constraints on the choice of a portfolio. One may, for instance, consider the following cases:

1. $\mathcal{H} = \mathbf{L}^a_{loc}(X)$: no constraints;
2. $\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : \quad H^i \ge 0, \quad 1 \le i \le m\}$: no short selling of the first m assets;
3. $\mathcal{H} = \left\{H \in \mathbf{L}^a_{loc}(X) : \quad \underline{G}^i \le H^i \le \overline{G}^i, \quad \underline{G}^i \le 0, \quad \overline{G}^i \ge 0, \quad 1 \le i \le d\right\}$ where $\underline{G}^i$ and $\overline{G}^i$ belong to $\mathbf{L}^a_{loc}(X)$: upper and lower bounds on the number of assets held in the portfolio.

The next theorem gives a "dual" characterization of $\mathcal{H}$-constrained portfolios. In the unconstrained case $\mathcal{H} = \mathbf{L}^a_{loc}(X)$, the theorem reduces to the optional decomposition theorem of Kramkov [14] and, in the case of a diffusion process X, of El Karoui and Quenez [7]; see also [10], [4], [13]. Consider the family of semimartingales

$$\mathcal{S} = \{H \bullet X : \quad H \in \mathcal{H}\}. \tag{4.2}$$

Theorem 4.1 *Let $\mathcal{P}(\mathcal{S}) \neq \emptyset$, and consider a process V which is locally bounded from below. Then the following statements are equivalent:*

(i) V is the value process of an $\mathcal{H}$-constrained portfolio, i.e.

$$V = V_0 + H \bullet X - C$$

with $H \in \mathcal{H}$ and an increasing process C

(ii) for all $Q \in \mathcal{P}(\mathcal{S})$ the process $V - A^{\mathcal{S}}(Q)$ is a local supermartingale under Q.

Proof In view of Theorem 3.1 the only delicate point in the proof is to check that the family $\mathcal{S}$ given by (4.2) satisfies Assumption 3.1. This follows from Mémin's theorem [17], which states that the space of stochastic integrals is closed in the semimartingale topology. $\square$

Following Cvitanić and Karatzas [2] and [3] let us also consider the case when constraints are imposed on the proportions of portfolio capital invested in the different stocks. To avoid technicalities we assume hereafter that X is a strictly positive process. Let

$$R_t^i = \int_0^t \frac{dX_s^i}{X_{s-}^i}, \quad t \geq 0,$$

denote the *return* process of the ith stock.

A portfolio Π is called *admissible* if it has a nonnegative value at any time instant. The value process of such a portfolio given by the additive representation (4.1) can also be written in the following multiplicative form:

$$V = V_0 \mathcal{E}\left(K \bullet R - D\right),$$

where $K_t^i = H_t^i X_{t-}^i / V_{t-} I_{\{V_{t-}>0\}}$ is the *proportion* of the portfolio capital invested in the ith stock at time t and

$$D_t = \int_0^t \frac{dC_s}{V_{s-}} I_{\{V_{s-}>0\}}, \quad t \geq 0,$$

is the accumulated proportion consumed up to time t.

Let now $\mathcal{K}$ be a family of integrands for R. As before we suppose that $\mathcal{K}$ contains the constant process $K \equiv 0$, is closed in $\mathbf{L}_{loc}^a$ with respect to the distance d_X, and is convex in the following sense: for any K and L in $\mathcal{K}$ and any predictable process $0 \leq h \leq 1$ the process $hK + (1-h)L$ belongs to $\mathcal{K}$.

The next theorem gives a "dual" characterization of admissible portfolios whose proportions take values in the set $\mathcal{K}$. Consider the family of semimartingales

$$\mathcal{S} = \{K \bullet R : \quad K \in \mathcal{K}\}, \tag{4.3}$$

and recall that $\mathcal{E}(X)$ denotes the Doléan-Dade exponential of the semimartingale X.

Theorem 4.2 *Let $\mathcal{P}(\mathcal{S}) \neq \emptyset$, and let V be a nonnegative process. Then the following statements are equivalent:*

(i) V is the value process of an admissible portfolio whose proportions belongs to $\mathcal{K}$, i.e.,

$$V = V_0 \mathcal{E}\left(K \bullet R - D\right)$$

with $K \in \mathcal{K}$ and an increasing process D

(ii) for all $Q \in \mathcal{P}(\mathcal{S})$ the process $V/\mathcal{E}(A^{\mathcal{S}}(Q))$ is a supermartingale under Q.

Proof The proof follows from Corollary 3.1 with the same arguments as in the proof of Theorem 4.1. $\square$

As an application let us consider the problem of (super-) replication of contingent claims with constrained portfolios. A contingent claim of European type is defined as a positive random variable f_T on $(\Omega, \mathcal{F}_T)$ interpreted as the value of the claim at time T. A strategy Π with value process V is called a *hedging* portfolio for the claim f_T if Π is admissible and $V_T \geq f_T$. An $\mathcal{H}$-constrained strategy $\widehat{\Pi}$ with value process $\widehat{V}$ is called a *minimal* $\mathcal{H}$-constrained hedging portfolio if

$$V_t \geq \widehat{V}_t \geq f_T I_{\{t \geq T\}}, \quad t \leq T,$$

for any $\mathcal{H}$-constrained hedging portfolio Π with value process V.

Proposition 4.1 *Let the set $\mathcal{S}$ be defined by (4.2). Assume that*

$$\sup_{Q \in \mathcal{P}(\mathcal{S})} E_Q(f_T - A^{\mathcal{S}}(Q)_T) < +\infty.$$

Then a minimal $\mathcal{H}$-constrained hedging strategy $\widehat{\Pi} = (\widehat{v}, \widehat{H}, \widehat{C})$ exists, and its value at time $t \leq T$ equals

$$\widehat{V}_t = \widehat{v} + (\widehat{H} \bullet X)_t - \widehat{C}_t = \operatorname*{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S})} \left(E_Q \left[f_T - A^{\mathcal{S}}(Q)_T \mid \mathcal{F}_t \right] + A^{\mathcal{S}}(Q)_t \right)^+,$$

where $a^+ = \max(a, 0)$.

Proposition 4.1 follows from Proposition 4.2 below which characterizes the value process of a minimal $\mathcal{H}$-constrained hedging portfolio for a contingent claim of American type.

Let $f = (f_t)_{t \geq 0}$ be a nonnegative process. We interpret f as the reward process of an American option. Note that if $f_t = f_T I_{\{t \geq T\}}$ then we have a contingent claim of European type. A portfolio Π with value process $V = (V_t)_{t \geq 0}$ is called a *hedging* strategy for f if

$$V_t \geq f_t, \quad t \geq 0.$$

An $\mathcal{H}$-constrained portfolio $\widetilde{\Pi}$ with value process $\widetilde{V} = (\widetilde{V}_t)_{t \geq 0}$ is called a *minimal* $\mathcal{H}$-constrained hedging portfolio if

$$V_t \geq \widetilde{V}_t \geq f_t, \quad t \geq 0$$

for any $\mathcal{H}$-constrained hedging portfolio Π with value process V.

The following theorem can be considered as a generalization of results of Bensoussan [1] and Karatzas [12] to the setting of markets with constraints.

For $Q \in \mathcal{P}(\mathcal{S})$ and $t \geq 0$ we denote by $\mathcal{M}_t(Q)$ the set of stopping times τ with values in $[t, +\infty)$ such that the process $\left(A^{\mathcal{S}}_{u \vee t}(Q) - A^{\mathcal{S}}_t(Q) \right)_{u \geq 0}$ is bounded on $[0, \tau]$.

Proposition 4.2 *Let the set $\mathcal{S}$ be defined by (4.2). Assume that*

$$\sup_{Q \in \mathcal{P}(\mathcal{S})} \sup_{\tau \in \mathcal{M}_0(Q)} E_Q(f_\tau - A^{\mathcal{S}}(Q)_\tau) < +\infty.$$

Then a minimal $\mathcal{H}$-constrained hedging portfolio $\widetilde{\Pi} = (\widetilde{v}, \widetilde{H}, \widetilde{C})$ exists, and its value at time $t \geq 0$ equals

$$\widetilde{V}_t = \widetilde{v} + (\widetilde{H} \bullet X)_t - \widetilde{C}_t = \operatorname*{ess\,sup}_{Q \in \mathcal{P}(\mathcal{S}), \tau \in \mathcal{M}_t(Q)} \left(E_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau | \mathcal{F}_t] + A^{\mathcal{S}}(Q)_t \right).$$

Proof Define

$$\widetilde{V}_t = \operatorname*{ess\,sup}_{Q\in\mathcal{P}(\mathcal{S}),\ \tau\in\mathcal{M}_t(Q)} \left(E_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau|\mathcal{F}_t] + A^{\mathcal{S}}(Q)_t\right).$$

Let $Q\in\mathcal{P}(\mathcal{S})$ and $(V_t)_{t\geq 0}$ be the value process of an $\mathcal{H}$-constrained hedging strategy. Let $(\tau_n)_{n\geq 1}$ be a localizing sequence such that $E_Q A^{\mathcal{S}}(Q)_{\tau_n}\leq n$. Since $V\geq f\geq 0$ we deduce from Theorem 4.1 that $V - A^{\mathcal{S}}(Q)$ is a supermartingale under Q on $[0,\tau_n]$. Therefore for any $t\geq 0$ and stopping time $\tau\in\mathcal{M}_t(Q)$ we have

$$\begin{aligned} V_{t\wedge\tau_n} &\geq E_Q[V_{\tau\wedge\tau_n} - A^{\mathcal{S}}(Q)_{\tau\wedge\tau_n}|\mathcal{F}_{t\wedge\tau_n}] + A^{\mathcal{S}}(Q)_{t\wedge\tau_n} \\ &\geq f_{\tau_n} I_{\{t>\tau_n\}} + E_Q\left[\left(f_{\tau\wedge\tau_n} - A^{\mathcal{S}}(Q)_{\tau\wedge\tau_n} + A^{\mathcal{S}}(Q)_t\right) I_{\{t\leq\tau_n\}}|\mathcal{F}_t\right]. \end{aligned}$$

¿From the definition of $\mathcal{M}_t(Q)$ we deduce that the sequence

$$\left(f_{\tau\wedge\tau_n} - A^{\mathcal{S}}(Q)_{\tau\wedge\tau_n} + A^{\mathcal{S}}(Q)_t\right) I_{\{t\leq\tau_n\}}, \quad n\geq 1,$$

is uniformly bounded from below. It follows from Fatou's lemma that

$$V_t \geq E_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau|\mathcal{F}_t] + A^{\mathcal{S}}(Q)_t,$$

hence

$$V_t\geq \widetilde{V}_t, \quad t\geq 0.$$

Note that $\widetilde{V}_t\geq f_t, \quad t\geq 0$. Therefore we only have to show that $\widetilde{V}$ is the value process of an $\mathcal{H}$-constrained portfolio. This fact follows from Theorem 4.1 and Lemma A.1 in the Appendix. $\square$

Similarly one may describe the value process of a minimal hedging portfolio whose proportions belong to the set $\mathcal{K}$.

Proposition 4.3 *Consider a contingent claim of European type given by $f_T\geq 0$, and let the set $\mathcal{S}$ be defined by (4.3). Assume that*

$$\sup_{Q\in\mathcal{P}(\mathcal{S})} E_Q\left[f_T/\mathcal{E}(A^{\mathcal{S}}(Q))_T\right] < +\infty.$$

Then there exists a hedging strategy $\widehat{\Pi}$ with proportions belonging to the set $\mathcal{K}$ such that its value at time t equals

$$\widehat{V}_t = \operatorname*{ess\,sup}_{Q\in\mathcal{P}(\mathcal{S})} \left(\mathcal{E}(A^{\mathcal{S}}(Q))_t E_Q\left[f_T/\mathcal{E}(A^{\mathcal{S}}(Q))_T \mid \mathcal{F}_t\right]\right).$$

Moreover, if V is the value of a hedging strategy with proportions in $\mathcal{K}$ then $V_t\geq\widehat{V}_t$, $t\geq 0$.

Denote by $\mathcal{M}_t$ the set of stopping times τ with values in $[t,+\infty)$.

Proposition 4.4 *Let $(f_t)_{t\geq 0}$ be a nonnegative process, and let the set $\mathcal{S}$ be defined by (4.3). Assume that*

$$\sup_{\tau\in\mathcal{M}_0}\sup_{Q\in\mathcal{P}(\mathcal{S})} E_Q\left[f_\tau/\mathcal{E}(A^{\mathcal{S}}(Q))_\tau\right] < +\infty.$$

Then there exists a hedging strategy $\widetilde{\Pi}$ with proportions belonging to the set $\mathcal{K}$ such that its value at time t equals

$$\widetilde{V}_t = \operatorname*{ess\,sup}_{Q\in\mathcal{P}(\mathcal{S}),\ \tau\in\mathcal{M}_t}\left[\mathcal{E}(A^{\mathcal{S}}(Q))_t E_Q[f_\tau/\mathcal{E}(A^{\mathcal{S}}(Q))_\tau|\mathcal{F}_t]\right].$$

Moreover, if V is the value of a hedging strategy with proportions in $\mathcal{K}$ then $V_t \geq \widetilde{V}_t$, $t\geq 0$.

The proofs are similar to the proof of Proposition 4.2 and are omitted here.

5 Fatou convergence

Let $(f^n)_{n\geq 1}$ be a sequence of measurable functions on $(\Omega,\mathcal{F},P)$. We use the standard notation $\mathbf{L}^0$ (resp. $\mathbf{L}^1$, $\mathbf{L}^\infty$) for the space of all (resp. P-integrable, P-essentially bounded) real-valued random variables on $(\Omega,\mathcal{F},P)$. If $\mathcal{C}$ is a subset in a linear space, then $\operatorname{conv}\mathcal{C}$ will denote the minimal convex set containing $\mathcal{C}$.

The work of McBeth [16], Schachermayer [19] and Delbaen and Schachermayer [5] has shown the usefulness of the following concept.

Definition 5.1 The sequence $(f^n)_{n\geq 1}$ is *Fatou convergent* to f if $(f^n)_{n\geq 1}$ is uniformly bounded from below and $f^n\to f$ almost surely. A subset $\mathcal{C}$ in $\mathbf{L}^0$ which is closed with respect to Fatou convergence will be called *Fatou closed.*

The following lemma on Fatou convergence is taken from [5], see also [19].

Lemma 5.1 *Let $(f^n)_{n\geq 1}$ be a sequence of nonnegative measurable functions.*

1) There is a sequence $g^n\in\operatorname{conv}(f^n,f^{n+1},\ldots)$, $n\geq 1$, which converges almost surely to a function g with values in $[0,\infty]$.

2) If $\operatorname{conv}(f^1,f^2,\ldots)$ is bounded in $\mathbf{L}^0$ then g is finite almost surely.

3) If there are $\alpha>0$ and $\delta>0$ such that $P(f^n>\alpha)>\delta$ for all n, then $P(g>0)>0$.

We will also need the following fact from functional analysis.

Theorem 5.1 *A convex set $\mathcal{C}$ in $\mathbf{L}^\infty$ is $\sigma(\mathbf{L}^\infty,\mathbf{L}^1)$-closed if and only if for each sequence $(f^n)_{n\geq 1}$ in $\mathcal{C}$, which is uniformly bounded and converges in probability to a function f, we have $f\in\mathcal{C}$.*

Let $\mathcal{C}$ be a convex set in $\mathbf{L}^0$ of functions which are bounded from below, and assume that $\mathcal{C}$ contains all bounded negative functions. We use the notation

$$\begin{aligned} a^{\mathcal{C}}(Q) &= \sup_{h\in\mathcal{C}} E_Q h,\\ \mathcal{P}(\mathcal{C}) &= \left\{Q\sim P:\quad a^{\mathcal{C}}(Q)<+\infty\right\}. \end{aligned}$$

Proposition 5.1 *Assume that the set $\mathcal{C}$ is Fatou closed, and that $\mathcal{P}(\mathcal{C}) \neq 0$. Then*

1) the set $\mathcal{C} \cap \mathbf{L}^\infty$ is $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed,

2) a function $g \in \mathbf{L}^0$ which is bounded from below belongs to $\mathcal{C}$ iff for all $Q \in \mathcal{P}(\mathcal{C})$

$$E_Q g \leq a^{\mathcal{C}}(Q). \tag{5.1}$$

Proof Let us define $C^\infty = C \cap L^\infty$. Note that

$$a^{\mathcal{C}}(Q) = \sup_{h \in \mathcal{C}^\infty} E_Q h,$$

since $\mathcal{C}$ is Fatou closed.

The assertion 1) follows immediately from Theorem 5.1 while the "only if" statement in 2) is trivial. To prove sufficiency in 2) let us first assume that g belongs to $\mathbf{L}^\infty$ and satisfies (5.1). Suppose that $g \notin \mathcal{C}^\infty$. Since $\mathcal{C}^\infty$ is convex and $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$-closed, we can apply the separation theorem to obtain a signed measure R with density in $\mathbf{L}^1$ such that

$$\sup_{h \in \mathcal{C}_0} E_R h < E_R g. \tag{5.2}$$

Since the set $\mathcal{C}^\infty$ contains all bounded negative random variables, R is a positive measure, which can be normalized to be a probability measure. If, in addition, $R \sim P$ then (5.2) implies that $R \in \mathcal{P}(\mathcal{C})$ and

$$a^{\mathcal{C}}(R) < E_R g,$$

in contradiction to (5.1).

In the general case where $R \ll P$, we define

$$R^\epsilon = \epsilon Q + (1 - \epsilon) R$$

for $Q \in \mathcal{P}(\mathcal{C})$ and $0 < \epsilon < 1$. It is clear that $R^\epsilon \in \mathcal{P}(\mathcal{C})$, and that

$$\sup_{h \in \mathcal{C}_0} E_{R^\epsilon} h = a^{\mathcal{C}}(R^\epsilon) < E_{R^\epsilon} g$$

if ϵ is small enough. So we only have to apply the preceding argument to R^ϵ instead of R in order to get a contradiction.

If g is only bounded from below then, as we have proved, the function $g^n = g \wedge n$ belongs to $\mathcal{C}^\infty$. This implies $g \in \mathcal{C}$ since the sequence $(g^n)_{n \geq 1}$ is Fatou convergent to g . $\square$

Now we introduce a concept of Fatou convergence in the setting of stochastic processes.

Definition 5.2 Let $\mathcal{T}$ be a dense subset of $\mathbf{R}_+$. A sequence of processes $(X^n)_{n \geq 1}$ is *Fatou convergent on* $\mathcal{T}$ to a process X if $(X^n)_{n \geq 1}$ is uniformly bounded from below, and if for any $t \geq 0$ we have

$$\begin{aligned} X_t &= \limsup_{s \downarrow t,\, s \in \mathcal{T}} \limsup_{n \to \infty} X_s^n \\ &= \liminf_{s \downarrow t,\, s \in \mathcal{T}} \liminf_{n \to \infty} X_s^n \end{aligned}$$

almost surely. If $\mathcal{T} = \mathbf{R}_+$ the sequence $(X^n)_{n \geq 1}$ is called simply *Fatou convergent.*

In analogy to Lemma 5.1 we have the following result.

Lemma 5.2 *1) Let $(X^n)_{n\geq 1}$ be a sequence of supermartingales which are uniformly bounded from below such that $X^n_0 = 0$, $n \geq 1$. Let $\mathcal{T}$ be a dense countable subset of $\mathbf{R}_+$. Then there is a sequence $Y^n \in \mathrm{conv}(X^n, X^{n+1}, \ldots)$, $n \geq 1$, and a supermartingale Y such that $Y_0 \leq 0$ and $(Y^n)_{n\geq 1}$ is Fatou convergent on $\mathcal{T}$ to Y.*

2) Let $(A^n)_{n\geq 1}$ be a sequence of increasing processes such that $A^n_0 = 0$, $n \geq 1$. There is a sequence $B^n \in \mathrm{conv}(A^n, A^{n+1}, \ldots)$, $n \geq 1$, and an increasing process B with values in $\overline{\mathbf{R}}_+$ such that $(B^n)_{n\geq 1}$ is Fatou convergent to B. If there are $T > 0$, $\alpha > 0$ and $\delta > 0$ such that $P(A^n_T > \alpha) > \delta$ for all $n \geq 1$, then $P(B_T > 0) > 0$.

Proof Assertion 2) was proved in [14]. To prove 1) we construct a sequence

$$Y^n \in \mathrm{conv}(X^n, X^{n+1}, \ldots), \quad n \geq 1,$$

such that $(Y^n_t)_{n\geq 1}$ converges almost surely to a variable Y'_t for all $t \in \mathcal{T}$. This can be done by Lemma 5.1 and the diagonal procedure of extracting a subsequence. Using Fatou's lemma we see that for $s < t$, $s \in \mathcal{T}$, $t \in \mathcal{T}$

$$E[Y'_t|\mathcal{F}_s] \leq \liminf_{n\to\infty} E[Y^n_t|\mathcal{F}_s] \leq \liminf_{n\to\infty} Y^n_s = Y'_s.$$

The standard construction based on Doob's Upcrossing lemma shows that the process Y defined as

$$Y_t = \lim_{s\downarrow t,\; s\in\mathcal{T}} Y'_s$$

is a right-continuous supermartingale with left limits. It is easy to see that $(Y^n)_{n\geq 1}$ is Fatou convergent to Y, and that

$$Y_0 \leq \liminf_{s\downarrow 0,\; s\in\mathcal{T}} \liminf_{n\to\infty} E[Y^n_s|\mathcal{F}_0] \leq 0.$$

□

Let now $\mathcal{X}$ be a predictably convex family of semimartingales which are locally bounded from below. We assume that $\mathcal{X}$ contains all locally bounded decreasing processes, in particular the process $X \equiv 0$. The following proposition will play a crucial role in the proof of Theorem 3.1.

Proposition 5.2 *Assume that $\mathcal{P}(\mathcal{X}) \neq \emptyset$, and that the set $\mathcal{X}$ is closed under Fatou convergence on some dense countable set $\mathcal{T} \subset \mathbf{R}_+$. Consider a stochastic process V which is locally bounded from below. Then V belongs to $\mathcal{X}$ if and only if for all $Q \in \mathcal{P}(\mathcal{X})$ the process $V - A^{\mathcal{X}}(Q)$ is a local supermartingale under Q.*

Proof The necessity follows from the definition of the upper variation process. The proof of sufficiency consists of two parts. First, we approximate V at a finite number of points by a process $X \in \mathcal{X}$. Then we pass to the limit and use the assumption that $\mathcal{X}$ is closed under

Fatou convergence in order to show that V belongs to $\mathcal{X}$. For simplicity we assume that $P \in \mathcal{P}(\mathcal{X})$, and that the upper variation process $A^{\mathcal{X}} = A^{\mathcal{X}}(P)$ and the process V satisfy

$$A^{\mathcal{X}} \leq N, \quad V_t - V_s \geq -N, \quad V_0 = 0, \tag{5.3}$$

for any $s < t$ and some $N \geq 0$. The general case will follow by a localization argument.

1) Let $\mathcal{T}'$ be a finite partition of $\mathbf{R}_+$. We are going to show the existence of $X \in \mathcal{X}$ which is bounded from below and such that

$$X_t = V_t \quad t \in \mathcal{T}'.$$

Note that one may choose X to be constant after the maximal point $t \in \mathcal{T}'$, in which case (5.3) and the supermartingale property of $X - A^{\mathcal{X}}$ imply that $X \geq -2N$. Since $\mathcal{X}$ is predictably convex, it is enough to show that

$$V_t - V_s = X_t - X_s \tag{5.4}$$

for any $s < t$ and some $X \in \mathcal{X}$ which is bounded from below. Hereafter we fix $s < t$ and denote $g = V_t - V_s$.

Let $\mathcal{Y}$ be the family of processes $Y \in \mathcal{X}$ which are bounded from below, equal to 0 on $[0, s]$, and constant on $[t, +\infty)$. Using the notation

$$\mathcal{C} = \{h| \quad h = Y_t, \quad Y \in \mathcal{Y}\},$$

our claim (5.4) means that $g \in \mathcal{C}$, and this will be deduced from Proposition 5.1.

First we show that $\mathcal{C}$ is Fatou closed. Let $(h^n)_{n\geq 1}$ be a sequence in $\mathcal{C}$ which is Fatou convergent to a function h, and let $(Y^n)_{n\geq 1}$ be a sequence in $\mathcal{Y}$ such that $Y^n_t = h^n$. Since $Y^n - A^{\mathcal{X}}$ is a supermartingale and the sequence $(Y^n_t - A^{\mathcal{X}}_t)_{n\geq 1}$ is uniformly bounded from below, the processes $(Y^n - A^{\mathcal{X}})_{n\geq 1}$ are uniformly bounded from below. ¿From Lemma 5.2 we deduce the existence of a sequence $Y^n \in \operatorname{conv}(X^n, X^{n+1}, \ldots)$, $n \geq 1$, and of a process Y such that $(Y^n)_{n\geq 1}$ is Fatou convergent to Y on the set $\mathcal{T}$. By assumption we have $Y \in \mathcal{Y}$, and this implies $h \in \mathcal{C}$ since

$$Y_t = \lim_{n\to\infty} Y^n_t = \lim_{n\to\infty} h^n = h.$$

To finish the proof of (5.4) we have to show that

$$E_Q g \leq \sup_{h\in\mathcal{C}} E_Q h \tag{5.5}$$

for any $Q \sim P$ such that the right-hand side of (5.5) is finite. Let Q be such a measure. Since

$$\sup_{h\in\mathcal{C}} E_Q h = \sup_{Y\in\mathcal{Y}} E_Q Y_t = \sup_{Y\in\mathcal{Y}} E_Q A^Y_t$$

we deduce from Lemma 2.1 that

$$\sup_{h\in\mathcal{C}} E_Q h = E_Q A^{\mathcal{Y}}(Q)_t, \tag{5.6}$$

where $A^{\mathcal{Y}}(Q)$ is the upper variation process of $\mathcal{Y}$ under Q. Note that $A^{\mathcal{Y}}(Q)$ is equal to 0 on $[0, s]$ and constant on $[t, +\infty)$.

Now let $R \sim P$ be a probability measure defined by the following properties:

1. $R = P$ on $\mathcal{F}_s$,
2. $E_R[\cdot|\mathcal{F}_s] = E_Q[\cdot|\mathcal{F}_s]$ on $\mathcal{F}_t$,
3. $E_R[\cdot|\mathcal{F}_t] = E_P[\cdot|\mathcal{F}_t]$ on $\mathcal{F}$.

We have $R \in \mathcal{P}(\mathcal{X})$ and

$$A^{\mathcal{X}}(R)_u = A^{\mathcal{X}}_{u\wedge s} + A^{\mathcal{X}}_{u\vee t} - A^{\mathcal{X}}_t + A^{\mathcal{Y}}(Q)_u, \quad u \geq 0.$$

The supermartingale property of $V - A^{\mathcal{X}}(R)$ under R implies

$$E_R[g|\mathcal{F}_s] = E_R[V_t - V_s|\mathcal{F}_s] \leq E_R[A^{\mathcal{X}}(R)_t - A^{\mathcal{X}}(R)_s|\mathcal{F}_s] = E_R[A^{\mathcal{Y}}_t(Q)|\mathcal{F}_s]$$

and therefore

$$E_Q[g|\mathcal{F}_s] \leq E_Q[A^{\mathcal{Y}}_t(Q)|\mathcal{F}_s],$$

which together with (5.6) implies (5.5).

2) Let $(\mathcal{T}_n)_{n\geq 1}$ be an increasing sequence of finite partitions such that $\mathcal{T} = \bigcup_{n\geq 1} \mathcal{T}_n$. As we have shown, there is a sequence $(X^n)_{n\geq 1}$ in $\mathcal{X}$ such that $X^n \geq -2N$ and $X^n_t = V_t$ for $t \in \mathcal{T}_n$. It follows that $(X^n)_{n\geq 1}$ is Fatou convergent on $\mathcal{T}$ to V. Since $\mathcal{X}$ is closed under Fatou convergence on $\mathcal{T}$, we conclude that $V \in \mathcal{X}$. $\square$

6 Proof of Theorem 3.1

Condition (ii) of the theorem is clearly necessary for a decomposition of the form (i). To prove sufficiency we denote by $\mathcal{X}$ the set of processes X which are locally bounded from below and are of the form $X = S - C$, where $S \in \mathcal{S}$ and C is an increasing process with $C_0 \geq 0$. Obviously, $\mathcal{P}(\mathcal{S}) = P(\mathcal{X})$. Thus, the proof follows from Proposition 5.2 and Proposition 6.1 below, which asserts that $\mathcal{X}$ is closed under Fatou convergence on dense subsets of $\mathbf{R}_+$.

Proposition 6.1 *Let $\mathcal{T}$ be a dense subset of $\mathbf{R}_+$. If a sequence $(X^n)_{n\geq 1}$ in $\mathcal{X}$ is Fatou convergent on $\mathcal{T}$ to X then $X \in \mathcal{X}$.*

Proof Clearly it is enough to show that X is dominated by some element $\widehat{X} \in \mathcal{X}$ with respect to the ordering $\prec$. Thus, the proposition follows from Lemma 6.1 and Lemma 6.3 below. $\square$

For a negative number a we denote $\mathcal{Y} = \mathcal{Y}(\mathcal{T}, a)$ the set of random processes Y such that there is a sequence $(Y^n)_{n\geq 1}$ in $\mathcal{X}$ which is Fatou convergent on $\mathcal{T}$ to Y and is bounded from below by the constant a. On $\mathcal{Y}$ we use the ordering $\prec$ defined by $X \prec Y$ iff $Y - X$ is an increasing process. To simplify the notation let us assume that $P \in \mathcal{P}(\mathcal{S})$, and let us write $A^{\mathcal{S}} = A^{\mathcal{S}}(P)$. In addition, we assume hereafter that all processes $X \in \mathcal{X}$ are constant after time 1 and that

$$A^{\mathcal{S}} \leq N$$

for some $N \geq 0$. The general case follows by a localization argument.

Lemma 6.1 *Let X be an element of $\mathcal{Y}$. There is a maximal element $\widehat{X}$ on the ordered set $\mathcal{Y}$ such that $X \prec \widehat{X}$.*

Proof For $Y \in \mathcal{Y}$ we define

$$b(Y) = \sup_{Z \in \mathcal{Y},\, Z \succ Y} E\left[Z_1 - Y_1\right].$$

It is clear that Y is maximal if and only if $b(Y) = 0$.

1) Let us show that, for any $Y \in \mathcal{Y}$ such that $b(Y) > 0$, there is $U \in \mathcal{X}$ such that $U \geq a$ and

$$P\left((U - Y)_1^* \geq 8\sqrt{b(Y)}\right) \leq 8\sqrt{b(Y)}, \tag{6.1}$$

where we use the notation

$$(U - Y)_1^* = \sup_{0 \leq t \leq 1} |U_t - Y_t|.$$

If this assertion fails, and if $(K^n)_{n \geq 1}$ is a sequence in $\mathcal{X}$ which is Fatou convergent on $\mathcal{T}$ to Y and such that $K^n \geq a$, $n \geq 1$, then

$$\liminf_{n \to \infty} P\left((K^n - Y)_1^* \geq 4\epsilon\right) \geq 4\epsilon, \tag{6.2}$$

where $\epsilon = 2\sqrt{b(Y)}$. In this case there are two increasing sequences $(i_k, j_k)_{k \geq 1}$ such that

$$P(\sup_{t \leq 1}(K_t^{i_k} - K_t^{j_k}) \geq \epsilon) \geq \epsilon, \quad k \geq 1.$$

Define the stopping time

$$\tau_k = \inf\left\{t \geq 0 : \quad K_t^{i_k} - K_t^{j_k} \geq \epsilon\right\},$$

and the processes

$$\begin{aligned} L_t^k &= K_{t \wedge \tau_k}^{i_k} + (K_t^{j_k} - K_{t \wedge \tau_k}^{j_k}), \\ A_t^k &= (K_{\tau_k}^{i_k} - K_{\tau_k}^{j_k}) I_{\{t \geq \tau_k\}}. \end{aligned}$$

It follows that $L^k \in \mathcal{X}$ and A^k is an increasing process such that $P(A_1^k \geq \epsilon) \geq \epsilon$. Moreover,

$$L_t^k - A_t^k = K_t^{i_k} I_{\{t < \tau_k\}} + K_t^{j_k} I_{\{t \geq \tau_k\}}.$$

Hence, the sequence $(L^k - A^k)_{k \geq 1}$ is Fatou convergent on $\mathcal{T}$ to Y and is bounded from below by the constant a.

Lemma 5.2 implies the existence of $B^k \in \mathrm{conv}(A^k, A^{k+1}, \ldots)$, $k \geq 1$, and of an increasing process B such that $(B^n)_{n \geq 1}$ is Fatou convergent on $\mathcal{T}$ to B. Since $P(A_1^k \geq \epsilon) \geq \epsilon$, we have that $EB_1 \geq \epsilon^2$. Denote by M^k the convex combination of $(L^k, L^{k+1}, \ldots)$ obtained with the same weights as B^k. The sequence $(M^n)_{n \geq 1}$ is Fatou convergent on $\mathcal{T}$ to $Y + B$ and $M^n \geq a$, $n \geq 1$. Therefore, $Z = Y + B$ belongs to $\mathcal{Y}$. It is clear that $Y \prec Z$. However

$$E\left[Z_1 - Y_1\right] = EB_1 \geq \epsilon^2 = 4b(Y)$$

and we come to a contradiction.

2) Let $(Y^n)_{n\geq 1}$ be a sequence in $\mathcal{Y}$ such that $Y^1 = X$, $Y^n \prec Y^{n+1}$ and $b(Y^n) \leq 2^{-2(n+3)}$, and let $\widehat{X}$ be the limit of $(Y^n)_{n\geq 1}$. ¿From 1) we deduce the existence of a sequence $(U^n)_{n\geq 1}$ in $\mathcal{X}$ such that $P\left((U^n - Y^n)_1^* \geq 2^{-n}\right) \leq 2^{-n}$ and $U^n \geq a$. It follows that $\widehat{X} \in \mathcal{Y}$. Finally, since $Y^n \prec \widehat{X}$ we have $X \prec \widehat{X}$ and $b(\widehat{X}) \leq \inf_{n\geq 1} b(Y^n) = 0$. $\square$

Lemma 6.2 *Let $\widehat{Y}$ be a maximal element of $\mathcal{Y}(\mathcal{T}, a)$. Let $(A^n)_{n\geq 1}$ be a sequence of increasing processes and $(K^n)_{n\geq 1}$ be a sequence in $\mathcal{S}$ such that $K^n \geq a$, $n \geq 1$. Assume that the convergence*

$$\begin{aligned}\widehat{Y}_t &= \limsup_{s\downarrow t,\, s\in\mathcal{T}} \limsup_{n\to\infty} (K^n_s - A^n_s) \\ &= \liminf_{s\downarrow t,\, s\in\mathcal{T}} \liminf_{n\to\infty} (K^n_s - A^n_s)\end{aligned}$$

holds almost surely, for any $t \geq 0$. Then the variables A^n_1 and the maximal functions $(K^n - \widehat{Y})^*_1$ tend to 0 in probability as n tends to ∞.

Proof If there are an increasing sequence $(n_k)_{k\geq 1}$ and a number $\epsilon > 0$ such that $P(A_1^{n_k} > \epsilon) > \epsilon$, then Lemma 5.2 implies the existence of $B^k \in \mathrm{conv}(A^{n_k}, A^{n_{k+1}}, \ldots)$, $k \geq 1$, and of an increasing process B such that $(B^n)_{n\geq 1}$ is Fatou convergent to B and $P(B_1 > 0) > 0$. Denote by N^k the convex combination of $(K^{n_k}, K^{n_{k+1}}, \ldots)$ obtained with the same weights as B^k. The sequence $(N^n)_{n\geq 1}$ is Fatou convergent on $\mathcal{T}$ to $\widehat{Y} + B$, and $N^n \geq a$. This contradicts the maximality of $\widehat{Y}$.

To finish the proof we have to show that the maximal functions $(K^m - K^n)^*_1$ tend to 0 in probability as m and n tend to ∞. This follows as in part 1) of the proof of Lemma 6.1. $\square$

The next lemma is the very assertion we need to finish the proof of Proposition 6.1.

Lemma 6.3 *Let $\widehat{Y}$ be a maximal element of $\mathcal{Y}(\mathcal{T}, a)$. Then $\widehat{Y} \in \mathcal{S}$.*

Proof The basic idea is to construct a sequence $(M^n)_{n\geq 1}$ in $\mathcal{S}$ which is convergent to $\widehat{Y}$ in the semimartingale topology and is uniformly bounded from below.

1) From Lemma 6.2 we deduce the existence of a sequence $(H^n)_{n\geq 1}$ in $\mathcal{S}$ such that $H^n \geq a$ and the maximal functions $(H^n - \widehat{Y})^*_1$ tend to 0 in probability. In particular, $\sup_{n\geq 1}(H^n)^*_1 < +\infty$. We are going to construct a sequence $(L^n)_{n\geq 1}$ of convex combinations of $(H^n)_{n\geq 1}$ which satisfies in addition the condition $\sup_{n\geq 1}[L^n, L^n]_1 < +\infty$.

The process $H^n - A^{\mathcal{S}}$ is a supermartingale and therefore can be decomposed as

$$H^n - A^{\mathcal{S}} = R^n - A^n,$$

where R^n is a local martingale and A^n is an increasing predictable process with $A^n_0 = 0$. Since $H^n \geq a$ and $A^{\mathcal{S}} \leq N$, we have $R^n \geq a - N$ and $A^n \leq R^n + N - a$. It follows that R^n is a supermartingale and $EA^n_1 \leq N - a$.

¿From Lemma 5.2 we deduce the existence of a sequence $B^n \in \mathrm{conv}(A^n, A^{n+1}, \ldots)$, $n \geq 1$, and of an increasing process B such that $(B^n)_{n\geq 1}$ is Fatou convergent to B. Fatou's lemma implies that $EB_1 \leq N - a$. It follows that

$$\sup_{n\geq 1} B_1^n < +\infty \quad \text{and} \quad \sup_{n\geq 1}[B^n, B^n]_1 = \sup_{n\geq 1} \sum_{t\geq 0} (\Delta B_t^n)^2 \leq \sup_{n\geq 1}(B_1^n)^2 < +\infty.$$

Let $K^n \in \mathrm{conv}(H^n, H^{n+1}, \ldots)$ and $S^n \in \mathrm{conv}(R^n, R^{n+1}, \ldots)$ be the convex combinations obtained with the same weights as $B^n \in \mathrm{conv}(A^n, A^{n+1}, \ldots)$. Since

$$K^n - A^{\mathcal{S}} = S^n - B^n,$$

we have

$$\sup_{n\geq 1}(S^n)_1^* \leq \sup_{n\geq 1}(K^n)_1^* + \sup_{n\geq 1} B_1^n + A_1^{\mathcal{S}} \leq \sup_{n\geq 1}(H^n)_1^* + \sup_{n\geq 1} B_1^n + A_1^{\mathcal{S}} < +\infty,$$

where

$$(S)_1^* = \sup_{t\leq 1} |S_t|.$$

It follows that the stopping time

$$\sigma_m = \inf_{n\geq 1} \inf \{0 \leq t \leq 1 : \quad |S_t^n| \geq m\}$$

is less then 1 with vanishing probability as m tends to ∞. In view of the supermartingale property of S^n and the inequality $S^n \geq a - N$, we obtain

$$E(S^n)_{\sigma_m}^* \leq m + E|S_{\sigma_m}^n| \leq m + 2N - 2a + ES_{\sigma_m}^n \leq m + 2N - 2a.$$

Now the Davis inequality implies the existence of a constant $c_m < +\infty$ such that

$$E[S^n, S^n]_{\sigma_m}^{1/2} \leq c_m.$$

¿From Lemma 5.2 we deduce the existence of a sequence

$$C^n \in \mathrm{conv}\left([S^n, S^n]^{1/2}, \left[S^{n+1}, S^{n+1}\right]^{1/2}, \ldots \right), \quad n \geq 1,$$

which is Fatou convergent to an increasing process C. An application of Fatou's lemma gives

$$EC_1 I_{\{\sigma_m = 1\}} \leq \liminf_{n\to\infty} EC_1^n I_{\{\sigma_m=1\}} \leq c_m.$$

Since $P(\sigma_m = 1)$ tends to 1 as m tends to ∞, we have $C_1 < +\infty$ almost surely. It follows that $\sup_{n\geq 1} C_1^n < +\infty$.

Let $L^n \in \mathrm{conv}(K^n, K^{n+1}, \ldots)$ and $T^n \in \mathrm{conv}(S^n, S^{n+1}, \ldots)$ and $D^n \in \mathrm{conv}(B^n, B^{n+1}, \ldots)$ be the convex combinations obtained with the same weights as C^n. We have

$$L^n = T^n + A^{\mathcal{S}} - D^n$$

and

$$[T^n, T^n]_1^{1/2} \leq C_1^n, \quad [D^n, D^n]_1^{1/2} \leq \sup_{k \leq n} [B^k, B^k]_1^{1/2};$$

see the "Minkowski inequality"(54.1) in [6], Chapter VII. Hence

$$\sup_{n \geq 1} [L^n, L^n]_1 \leq 3 \sup_{n \geq 1} [T^n, T^n]_1 + 3 \sup_{n \geq 1} [D^n, D^n]_1 + 3[A^{\mathcal{S}}, A^{\mathcal{S}}]_1 < +\infty.$$

2) Let us define the probability measure $R \sim P$ on $(\Omega, \mathcal{F})$ with density

$$\frac{dR}{dP} = \frac{e^{-\rho}}{Ee^{-\rho}},$$

where $\rho = \sup_{n \geq 1} [L^n, L^n]_1$. Since

$$E_R \sup_{n \geq 1} [L^n, L^n]_1 < +\infty,$$

L^n is a special semimartingale with respect to R, and so it can be decomposed as

$$L^n = F^n + A^n,$$

where F^n is an local martingale under R and A^n is a predictable process of bounded variation. Note that

$$E_R [F^n, F^n]_1 \leq E_R [L^n, L^n]_1 ;$$

see [6], VII Theorem 55. This implies $\sup_{n \geq 1} E_R [F^n, F^n]_1 < +\infty$, i.e., the sequence $(F^n)_{n \geq 1}$ is bounded in the space $\mathcal{M}^2(R, [0,1])$ of square integrable martingales with respect to R and with parameter set $[0,1]$. Thus there is a sequence $G^n \in \mathrm{conv}(F^n, F^{n+1}, \ldots)$, $n \geq 1$, which is convergent in $\mathcal{M}^2(R, [0,1])$, hence also in the semimartingale topology.

3) Let $(M^n)_{n \geq 1}$ be the sequence of convex combinations of $(L^n)_{n \geq 1}$ with the canonical decomposition

$$M^n = G^n + B^n,$$

where G^n is the convergent sequence of martingales constructed in 2) and $B^n \in \mathrm{conv}(A^n, A^{n+1}, \ldots)$. In order to establish convergence of $(M^n)_{n \geq 1}$ in the semimartingale topology we only have to show that the sequence $(B^n)_{n \geq 1}$ is convergent in the semimartingale topology. The proof proceeds along the lines of the proof of Lemma 4.11 in [5]. It is sufficient to show that $\int_0^1 |dB_t^n - dB_t^m|$ tends to 0 as n and m tend to ∞. If this were not the case, we could find two increasing sequences $(i_n, j_n)_{n \geq 1}$ and a number $\epsilon > 0$ such that $P(C_1^n > \epsilon) > \epsilon$, where

$$C_t^n = \frac{1}{2} \int_0^t |dB_s^{i_n} - dB_s^{j_n}|, \quad t \geq 0.$$

Let us show that this contradicts the maximal property of $\widehat{Y}$.

Hahn's decomposition implies the existence of a predictable process h^n with values in $\{-1, 1\}$ such that

$$C_t^n = \frac{1}{2} \int_0^t h^n (dB^{i_n} - dB^{j_n}), \quad t \geq 0.$$

Let us denote

$$N_t^n = \frac{1}{2}\int_0^t (1+h^n)dM^{i_n} + \frac{1}{2}\int_0^t (1-h^n)dM^{j_n}$$

and let $N^n = H^n + D^n$ be the canonical decomposition of N^n under R, where

$$H_t^n = \frac{1}{2}\int_0^t (1+h^n)dG^{i_n} + \frac{1}{2}\int_0^t (1-h^n)dG^{j_n}$$

is a martingale under R and

$$D_t^n = \frac{1}{2}\int_0^t (1+h^n)dB^{i_n} + \frac{1}{2}\int_0^t (1-h^n)dB^{j_n}$$

is a predictable process of bounded variation. Since $M^n \in \mathcal{S}$ and the set $\mathcal{S}$ is predictably convex we have $N^n \in \mathcal{S}$.

By the construction of h^n we deduce that the processes $D^n - B^{i_n}$ and $D^n - B^{j_n}$ are increasing. Moreover, since

$$H_t^n - G_t^{i_n} = \frac{1}{2}\int_0^t (h^n - 1)\left(dG^{i_n} - dG^{j_n}\right)$$

and the processes $G^{i_n} - G^{j_n}$ tend to 0 in $\mathcal{M}^2(R,[0,1])$, the maximal functions $(H^n - G^{i_n})_1^*$ tend to 0 in probability. The same holds for $(H^n - G^{j_n})_1^*$. Taking if necessary a subsequence we can suppose that convergence holds almost surely and that the stopping times

$$\tau_n = \inf_{k\geq n} \inf\left\{0 \leq t \leq 1 : \quad H^k < \max\left(G^{i_k}, G^{j_k}\right) - 1/n\right\}$$

are equal to 1 with probability tending to 1 as n tends to ∞. Since $(M^{i_n} - \widehat{Y})_1^* \to 0$ in probability and

$$N^n - C^n = H^n + \frac{1}{2}(B^{i_n} + B^{j_n}) = H^n - \frac{1}{2}(G^{i_n} + G^{j_n}) + \frac{1}{2}(M^{i_n} + M^{j_n})$$

we deduce that the maximal functions $(N^n - C^n - \widehat{Y})_1^*$ tend to 0 in probability.

For $t < \tau_n$ and $k \geq n$ we have

$$\begin{aligned} N_t^k = H_t^k + D_t^k &\geq \max(G_t^{i_k}, G_t^{j_k}) + \max(B_t^{i_k}, B_t^{j_k}) - 1/n \\ &\geq \max(M_t^{i_k}, M_t^{j_k}) - 1/n. \end{aligned}$$

At time τ_n a jump ΔN^k is either ΔM^{i_k} or ΔM^{j_k} and hence the equality

$$N_t^k \geq \min(M_t^{i_k}, M_t^{j_k}) - 1/n$$

holds for $t \leq \tau_n$. Since $M^n \geq a$ we get

$$N_t^k \geq a - 1/n, \quad 0 \leq t \leq \tau_n, \quad k \geq n.$$

If we define $\widehat{N}_t^n = \frac{na}{na-1} N_{t\wedge\tau_n}^n$, $t \geq 0$, then $\widehat{N}^n \in \mathcal{S}$, $\widehat{N}^n \geq a$ and maximal functions $(\widehat{N}^n - C^n - \widehat{Y})_1^*$ tend to 0 in probability.

Now Lemma 6.2 implies that the variables C_1^n tend to 0 in probability. This contradiction proves the convergence of $(B^n)_{n\geq 1}$ in the semimartingale topology. $\square$

Appendix: A stochastic control lemma

Here we prove a stochastic control lemma which was used in the proof of Proposition 4.2. Let $\mathcal{S}$ be a family of semimartingales which are locally bounded from below. We suppose that $\mathcal{P}(\mathcal{S}) \neq \emptyset$ and denote by $A^{\mathcal{S}}(Q)$ the upper variation process of $\mathcal{S}$ with respect to $Q \in \mathcal{P}(\mathcal{S})$. Recall that $\mathcal{M}_t(Q)$ denotes the set of stopping times τ with values in $[t, +\infty[$ and such that the process $\left(A^{\mathcal{S}}(Q)_{u\vee t} - A^{\mathcal{S}}(Q)_t\right)_{u\geq 0}$ is bounded on $[0,\tau]$. As before, all processes are assumed to be real-valued, to have right-continuous paths with left limits, and to be adapted with respect to the given filtration $(\mathcal{F}_t)_{t\geq 0}$. For simplicity we assume hereafter that the initial σ-field $\mathcal{F}_0$ is trivial.

Lemma A.1 *Let $(f_t)_{t\geq 0}$ be a nonnegative process such that*

$$\sup_{Q\in\mathcal{P}(\mathcal{S})} \sup_{\tau\in\mathcal{M}_0(Q)} E_Q(f_\tau - A^{\mathcal{S}}(Q)_\tau) < +\infty.$$

There exists a process $(U_t)_{t\geq 0}$ such that for $t \geq 0$

$$U_t = \operatorname*{ess\,sup}_{Q\in\mathcal{P}(\mathcal{S}),\ \tau\in\mathcal{M}_t(Q)} \left(E_Q[f_\tau - A^{\mathcal{S}}(Q)_\tau|\mathcal{F}_t] + A^{\mathcal{S}}(Q)_t\right)$$

almost surely. Moreover, for any $Q \in \mathcal{P}(\mathcal{S})$ the process $U - A^{\mathcal{S}}(Q)$ is a local supermartingale under Q.

Proof Without loss of generality we can suppose that $P \in \mathcal{P}(\mathcal{S})$. For simplicity we assume that the upper variation process of $\mathcal{S}$ with respect to P is uniformly bounded, i.e.,

$$A^{\mathcal{S}} := A^{\mathcal{S}}(P) \leq N \tag{A.1}$$

for some $N < \infty$. The general case follows by localization arguments.

We need to show that U is a supermartingale with respect to P. To any $Q \in \mathcal{P}(\mathcal{S})$ we can associate the corresponding density process z with respect to P. For $t \geq 0$ we denote by $\mathcal{Z}_t$ the set of density processes z corresponding to some $Q \in \mathcal{P}(\mathcal{S})$ which are equal to 1 on the interval $[0,t]$. Throughout we will use the notation

$$A^{\mathcal{S}}(z) = A^{\mathcal{S}}(Q), \quad \mathcal{M}_t(z) = \mathcal{M}_t(Q)$$

if $z \in \mathcal{Z}_t$ corresponds to $Q \in \mathcal{P}(\mathcal{S})$. Due to the fact that $A^{\mathcal{S}}(z) = A^{\mathcal{S}}$ on $[0,t]$ for $z \in \mathcal{Z}_t$, we get

$$U_t = \operatorname*{ess\,sup}_{z\in\mathcal{Z}_t,\ \tau\in\mathcal{M}_t(z)} E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau + A^{\mathcal{S}}_t)|\mathcal{F}_t].$$

For $n \geq 1$ we denote by $\mathcal{M}_t(z,n)$ the set of stopping times τ with values in $[t, +\infty[$ and such that the process $\left(A^{\mathcal{S}}(Q)_{u\vee t} - A^{\mathcal{S}}(Q)_t\right)_{u\geq 0}$ is bounded by n on $[0,\tau]$. We also define the process

$$U^n_t = \operatorname*{ess\,sup}_{z\in\mathcal{Z}_t,\ \tau\in\mathcal{M}_t(z,n)} E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau + A^{\mathcal{S}}_t)|\mathcal{F}_t].$$

Since for any $z \in \mathcal{Z}_t$

(A.2) $$\mathcal{M}_t(z) = \bigcup_{n \geq 1} \mathcal{M}_t(z, n),$$

we deduce that

(A.3) $$U_t = \sup_{n \geq 1} U_t^n.$$

Let z_1 and z_2 belong to $\mathcal{Z}_t$, $\tau_1 \in \mathcal{M}_t(z_1, n)$ and $\tau_2 \in \mathcal{M}_t(z_2, n)$, where n is a fixed positive number. Define the set

$$K = \Big\{\omega : \quad E[z_{1\tau_1}(f_{\tau_1} - A^{\mathcal{S}}(z_1)_{\tau_1}|\mathcal{F}_t] > E[z_{2\tau_2}(f_{\tau_2} - A^{\mathcal{S}}(z_2)_{\tau_2}|\mathcal{F}_t]\Big\}.$$

Since $K \in \mathcal{F}_t$, we conclude that the process

$$z = z_1 I_K + z_2(1 - I_K)$$

belongs to $\mathcal{Z}_t$ and the stopping time

$$\tau = \tau_1 I_K + \tau_2(1 - I_K)$$

is an element of $\mathcal{M}(z, n)$. Moreover, we have

$$\begin{aligned} A^{\mathcal{S}}(z) &= A^{\mathcal{S}}(z_1) I_K + A^{\mathcal{S}}(z_2)(1 - I_K), \\ E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau)|\mathcal{F}_t] &= \max\Big\{E[z_{1\tau_1}(f_{\tau_1} - A^{\mathcal{S}}(z_1)_{\tau_1})|\mathcal{F}_t], E[z_{2\tau_2}(f_{\tau_2} - A^{\mathcal{S}}(z_2)_{\tau_2})|\mathcal{F}_t]\Big\}. \end{aligned}$$

Note that for any $z \in \mathcal{Z}_t$ and $\tau \in \mathcal{M}_t(z, n)$ we have

$$z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau + A_t^{\mathcal{S}} + n) \geq 0.$$

Now the results of Striebel [20] (see [8], Lemma 16.A.5) imply that for $s \leq t$:

$$E[U_t^n|\mathcal{F}_s] = \operatorname*{ess\,sup}_{z \in \mathcal{Z}_t,\ \tau \in \mathcal{M}_t(z,n)} E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau + A_t^{\mathcal{S}})|\mathcal{F}_s].$$

Taking the supremum over n and using (A.2) and (A.3) we get

(A.4) $$E[U_t|\mathcal{F}_s] = \operatorname*{ess\,sup}_{z \in \mathcal{Z}_t,\ \tau \in \mathcal{M}_t(z)} E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau + A_t^{\mathcal{S}})|\mathcal{F}_s].$$

Evidently, $\mathcal{Z}_t \subset \mathcal{Z}_s$. Moreover, we have $\mathcal{M}_t(Q) \subset \mathcal{M}_s(Q)$ for $Q \in \mathcal{P}(\mathcal{S})$ due to (A.1) . It follows that

$$\begin{aligned} E[U_t - A_t^{\mathcal{S}}|\mathcal{F}_s] &\leq \operatorname*{ess\,sup}_{z \in \mathcal{Z}_s,\ \tau \in \mathcal{M}_s(z)} E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau)|\mathcal{F}_s] \\ &= U_s - A_s^{\mathcal{S}}. \end{aligned}$$

Hence $U - A^{\mathcal{S}}$ is a supermartingale.

To finish the proof we have to show that U admits a right-continuous modification with limits from the left. This is equivalent to the existence of such a modification for $V = U - A^{\mathcal{S}}$.

According to Theorem 3.1 in [15], this is the case if and only if the function $(EV_t)_{t\geq 0}$ is right-continuous.

When $s = 0$ the equality (A.4) takes the form

$$EV_t = \sup_{z\in\mathcal{Z}_t,\ \tau\in\mathcal{M}_t(z)} E[z_\tau(f_\tau - A^{\mathcal{S}}(z)_\tau)]. \tag{A.5}$$

Let t, $(t_n)_{n\geq 1}$ be positive numbers such that $t_n \downarrow t$, $n \to +\infty$, and $t_n < t+1$, $n \geq 1$. Since V is a supermartingale, we have

$$EV_t \geq \lim_{n\to\infty} EV_{t_n}.$$

To prove the reverse inequality we fix $\epsilon > 0$ and choose a process $z = z(\epsilon)$ from $\mathcal{Z}_t$ and a stopping time $\sigma = \sigma(\epsilon)$ from $\mathcal{M}_t(z)$ such that

$$EV_t < E[z_\sigma(f_\sigma - A^{\mathcal{S}}(z)_\sigma)] + \epsilon \quad \text{and} \quad P(\sigma > t) = 1. \tag{A.6}$$

This is possible by (A.5) and the right-continuity of the processes under consideration. For $n \geq 1$ we define the stopping time σ_n and the process z^n as follows

$$\sigma_n = \begin{cases} \sigma, & \sigma \geq t_n \\ t+1, & \sigma < t_n \end{cases}, \qquad z^n_u = \begin{cases} z_u/z_{t_n}, & \sigma \geq t_n \text{ and } u \geq t_n \\ 1, & \sigma < t_n \text{ or } u < t_n \end{cases}.$$

We have that $z^n \in \mathcal{Z}_{t_n}$, $\sigma_n \in \mathcal{M}_{t_n}(z^n)$ and

$$A^{\mathcal{S}}(z^n)_u = \begin{cases} (A^{\mathcal{S}}(z)_u - A^{\mathcal{S}}(z)_{t_n} + A^{\mathcal{S}}_{t_n}), & \sigma \geq t_n \text{ and } u \geq t_n \\ A^{\mathcal{S}}_u, & \sigma < t_n \text{ or } u < t_n \end{cases}.$$

Since

$$A^{\mathcal{S}}(z^n)_{\sigma_n} \leq A^{\mathcal{S}}_{\sigma_n} + A^{\mathcal{S}}(z)_\sigma - A^{\mathcal{S}}(z)_t,$$

it follows from (A.1) and the definition of the set $\mathcal{M}_t(z)$ that the sequence $\left(A^{\mathcal{S}}(z^n)_{\sigma_n}\right)_{n\geq 1}$ is uniformly bounded. Now we use Fatou's lemma and (A.6) to conclude that

$$EV_t \leq \liminf_{n\to\infty} E[z^n_{\sigma_n}(f_{\sigma_n} - A^{\mathcal{S}}(z^n)_{\sigma_n})] + \epsilon \leq \lim_{n\to\infty} EV_{t_n} + \epsilon.$$

Hence, $(EV_t)_{t\geq 0}$ is a right-continuous function. This completes the proof of the lemma. $\square$

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